ON THE CHARACTERIZATION OF THE STRONG ENVELOP

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Abstract

On the infinite time interval $[0, \infty]$ we consider the class $[\mathfrak{B}]$ of bounded measurable processes and for each element $X \in [\mathfrak{B}]$ its strong envelop, that is the smallest supermartingale bounding X from above. We provide some properties and derive two important characterizations of strong envelop by using firstly the general Skorohod condition, and secondly we show that is the unique solution of the stochastic variational inequality (SVI). This leads to several a priori estimate.

Key words : Strong envelope; Snell envelope; Stochastic variational inequality (SVI).

MSC Classification: 60H30; 60G40; 93E20

1 Introduction and some preliminary proposition

Let $[0, \infty[$ be an infinite time interval and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Denote $[\mathfrak{B}]$ the class of bounded measurable processes, $[D]$ the class of progressively measurable càdlàg processes, S^2 the Banach space of all càdlàg process Y such that

$$\|Y\|_{S^2} := \left(\mathbb{E}\left[\sup_{t \geq 0} Y_t\right]^2\right)^{\frac{1}{2}} < \infty. \tag{1.2}$$

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and H^2 the space of semimartingales Y such that

$$\|Y\|_{H^2} := \inf_{Y=M+A} \left(\left\| [m]_\infty^{1/2} + \int_0^\infty |dA_s| \right\|_{L^2} \right) < \infty. \tag{1.3}$$

It is well known from (Stettner and Zabczyk [9]) that for any $X \in [\mathfrak{B}]$ there exists a smallest supermartingale U bounding X from above a.e., a.s., which is called *strong envelop* of X and has the following property

$$U_t(\omega) \quad \geq \quad X_t(\omega), \quad dt \otimes dP \ a.e. \tag{1.4}$$

The main objective of this paper, is to provide a tools to characterize the strong envelope of the given process, using both, general Skorohod conditions and stochastic variational inequality (SVI).

Definition 1.2 *Strong envelop*
Let $(X_t)_{t\geq 0}$ be a bounded progressively measurable process such that $\mathbb{E}\int_0^{+\infty}|X_s|ds < \infty$. A right-continuous supermartingale $U := (U_t)_{t\geq 0}$ is called the strong envelope of $X := (X_t)_{t\geq 0}$ if it is the smallest right-continuous, non-negative super-martingale such that $U \geq X$, a.s. a.e.; i.e., if $(\bar{U}_t)_{t\geq 0}$ another supermartingale such that for all $t \geq 0$, $\bar{U}_t \geq X_t$, $dt \otimes dP$, then $\bar{U}_t \geq U_t$, $dt \otimes dP$ for any $t \geq 0$ and we write $U = SE(X)$.

We formulate now the problem of strong envelop (SE in short):
For an arbitrary $\beta > 0$ find a right-continuous process U^β such that for all $t \geq 0$

$$U_t^\beta := \beta \mathbb{E}\left(\int_t^{+\infty} (X_s - U_s^\beta)^+ ds / \mathcal{F}_t \right) \quad P\text{-a.s.} \tag{1.5}$$

We recall Theorem 9 in [9]

Theorem 1.2 *If X is a bounded, progressively measurable process such $\mathbb{E}\int_0^{+\infty}|X_s|ds < \infty$, then for each $\beta > 0$, there is a solution of (1.5) and it is unique, up to indistinguishable processes. It increases with $\beta \nearrow +\infty$ and the limit process*

$$U_t := \lim_{\beta\to\infty} U_t^\beta, \qquad 0 \leq t \leq T, \tag{1.6}$$

is the strong envelop of (X_t).

We give in the following Proposition as the main convergence property of strong envelop.

Proposition 1.2 *The process $U := SE(X)$ enjoys the following properties:*

(i) The Doob-Meyer decomposition of the right-continuous supermartingale U implies the existence of a martingale $(M_t)_{t\geq 0}$ and a non-decreasing processes $(A_t)_{t\geq 0}$ which is right-continuous and predictable such that,

$$U_t = M_t - A_t, \quad \forall t \geq 0.$$

(ii) If $(X^n)_{n\geq 0}$ and X are a bounded progressively measurable processes such that the sequence $(X^n)_{n\geq 0}$ converges increasingly to X, then $SE(X^n)_{n\geq 0}$ converges increasingly to $SE(X)$.

Proof. The first and third assertion are obvious, we then prove (ii).
Since X^n converges increasingly to X, it follows that for all $t \geq 0$, $SE(X^n) \leq SE(X)$ a.s., a.e.. Furthermore,

$$\lim_{n\to\infty} SE(X^n) \quad \leq \quad SE(X) \text{ a.s., a.e.} \tag{1.7}$$

Then $\lim_{n\to\infty} SE(X^n)$ is càdlàg supermartingale (see e.g., Delacherie and Meyer [2] p. 86). From other side, $X^n \leq SE(X^n)$ as., a.e., implies that $dt \otimes d\mathbb{P}$ for all $t \in [0, T]$, $X_t \leq \lim_{n\to\infty} SE(X^n)$. However, since $SE(X)$ is the smallest supermartingale which dominate X then $SE(X) \leq \lim_{n\to\infty} SE(X^n)$ $dt \otimes d\mathbb{P}$, together with (1.7) we get the required result. □

1.1 Optimal stopping times

In the following we prove an important property of strong envelop, in the spirit of the central result in the theory of Snell envelop (see Shashtashvili [7], Theorem 1)

Theorem 1.3 *Let X be a bounded progressively measurable process. Then for the strong envelop U of X we have*

$$U_t = \mathbb{E}(U_{\tau_t^\varepsilon}/\mathcal{F}_t), \tag{1.8}$$

for each positive $\varepsilon > 0$, $t \geq 0$ and $\tau_t^\varepsilon = \inf\{s \geq t : X_s \geq U_s - \varepsilon\}$. Furthermore, τ_t^ε is ε-optimal stopping time after t, i.e.,

$$\mathbb{E}(X_{\tau_t^\varepsilon}) \quad \geq \quad \mathbb{E}(U_t) - \varepsilon.$$

Proof. We follow the same argument as the proof of Proposition 1.3. We consider the stoping time

$$\tau_t^\varepsilon = \inf\{s \geq t : X_s \geq U_s - \varepsilon\}.$$

We have for any $t \geq 0$ and $\varepsilon > 0$

$$\begin{aligned} \mathbb{E}\left[U_{\tau_t^\varepsilon} - U_t/\mathcal{F}_t\right] &= -\mathbb{E}\left[\int_0^{+\infty} 1_{t\leq s\leq \tau_t^\varepsilon} dA_s/\mathcal{F}_t\right] \\ &\leq 0. \end{aligned}$$

Observe that for all $X^* \in [D]$ such that $X \leq X^* \leq U$, we have

$$[t, \tau_t^\varepsilon[\quad \subset \quad \{s \geq 0/ \;\; U_{s^-} - X^*_{s^-} > 0\}, \tag{1.9}$$

which leads to

$$\mathbb{E}\left[\int 1_{t\leq s\leq \tau_t^\varepsilon} 1_{U_{s-} > X^*_{s-}} dA_s/\mathcal{F}_t\right] = 0.$$

At last we conclude that

$$U_t = \mathbb{E}\left[U_{\tau_t^\varepsilon}/\mathcal{F}_t\right].$$

From other side, we have

$$\mathbb{E}(X_{\tau_t^\varepsilon}) \;\geq\; \mathbb{E}(U_{\tau_t^\varepsilon}) - \varepsilon = \mathbb{E}(U_t) - \varepsilon.$$

□

Remark 1.2 *The nondecreasing process A in the Doob-Meyer decomposition of the strong envelope U, has the following property*

$$A_t = A_{\tau_t^\varepsilon}, \qquad P\text{-}a.s.$$

for $t \geq 0$ and and $\varepsilon > 0$.

1.2 Characterization of strong envelop

In this subsection we provide two important characterizations of strong envelop, using firstly the generalized notion of Skorohod reflecting conditions (see [8]), similar to the one was given by Peng and Xu [6] in backward stochastic differential equation (BSDE in short).

Proposition 1.3 *Let U be a super-martingale and X a bounded progressively measurable process, then we have the following equivalence:*

1. *U is the strong envelope of X.*
2. **a** *$U \geq X$, as. a.e.;*

   b *The following (generalized) Skorohod condition (cf. [8]) holds:*

$$\int_0^{+\infty} (U_{s^-} - X^*_{s^-})dA_s = 0, \quad a.s., \; \forall X^* \in D \; s.t\; U \geq X^* \geq X \quad a.s., \; a.e. \tag{1.10}$$

where A is the increasing process appearing in Doob-Meyer decomposition of U.

Proof.
We prove $1 \Longrightarrow 2$.

Let $U = SE(X)$, then (2.a) is obvious.

Let X^* be a càdlàg process such that $U \geq X^*$, then $U_- \geq X^*_-$ and $U \geq SN(X^*) \geq X$, where $SN(X^*)$ is the Snell envelop of X^*. However,U is the smallest super-martingale who major X then $U \leq SN(X^*)$, combining this with inverse inequality we have $U = SN(X^*)$. By the unique Doob-Meyer decomposition of $U_t = M_t - A_t$ and $SN(X^*_t) = M^*_t - A^*_t$, we have $M^*_t = M_t$ and $A^*_t = A_t$, leads to

$$\int_0^{\cdot} 1_{\{U_{t-} > X^*_{t-}\}} dA_s = \int_0^{\cdot} 1_{\{SN(X^*)_{t-} > X^*_{t-}\}} dA^*_s = 0.$$

To prove the inverse inequality, we consider a càdlàg process X^* such that $X \leq X^* \leq U \wedge SE(X)$, and let the stopping time $\tau_t^{\varepsilon *}$ be such that

$$\tau_t^{\varepsilon *} = \inf\{s \geq t, \text{ s.t. } \quad X_s^* \geq U_s - \varepsilon\}. \tag{1.11}$$

Then

$$\mathbb{E}\left[U_{\tau_t^{\varepsilon *}} - U_t/\mathcal{F}_t\right] \quad = \quad -\mathbb{E}\left[\int 1_{t \leq s \leq \tau_t^{\varepsilon}} dA_s/\mathcal{F}_t\right] \leq 0$$

Notice that $[t, \tau_t^{\varepsilon *}[\subset \ \{s \in [0,T]/ \ \ U_{s-} - X_{s-}^* > 0\}$, which leads to

$$\mathbb{E}\left[\int 1_{t \leq s \leq \tau_t^{\varepsilon *}} 1_{U_{s-} > X_{s-}^*} dA_s/\mathcal{F}_t\right] = 0.$$

Combining this with the latest above inequality, we conclude the martingale property of U and we have

$$U_t = \mathbb{E}\left[U_{\tau_t^{\varepsilon *}}/\mathcal{F}_t\right].$$

From other side, $U_{\tau_t^{\varepsilon *}} \leq X_{\tau_t^{\varepsilon}}^* + \varepsilon$ taking the conditional expectation in both side

$$\begin{aligned} U_t &\leq \mathbb{E}\left[X_{\tau_t^{\varepsilon *}}^*/\mathcal{F}_t\right] + \varepsilon \\ &\leq SN(X^*)_t + \varepsilon. \end{aligned}$$

By sending ε to 0, we get $SN(X^*) = SE(X) = U$. □

Remark 1.3 *If we assume furthermore that X is a càdlàg process, then U is the snell envelop of X and Skorohod condition (1.10) become*

$$\int_0^{\infty} (U_{s^-} - X_{s^-}) dA_s = 0.$$

The second characterization of strong envelope is defined as the unique solution stochastic variational inequality (SVI in short) introduced in [1]. Let K be a convex subset of the space S^2, taking the following form

$$K = \{V \in S^2, \quad V_t \geq X_t, \ \forall t \geq 0, \ dt \otimes dP\}. \tag{1.12}$$

The problem of stochastic variational inequality associated to optimal stopping time, consists to find an element $U \in K \cap H^2$ such that for any element $V \in K$, a pair of stopping times (τ_1, τ_2) where $0 \leq \tau_1 \leq \tau_2$, the following inequality should hold

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} (U_{s-} - V_{s-}) dU_s/\mathcal{F}_{\tau_1}\right] \geq 0 \quad \text{ a.s.} \tag{1.13}$$

Theorem 1.4 *There exist a unique solution U of the SVI (1.13), which is the strong envelop of X.*

Proof. Let us check that $U = SE(X)$ is a solution of above SVI.
From Proposition 1.2

$$U_t = M_t - A_t.$$

Taking $V \in K$, we have $U \leq SN(V)$. It follows that

$$\begin{aligned}\mathbb{E}\left[\int_{\tau_1}^{\tau_2} [U_{s-} - V_{s-}] dU_s / \mathcal{F}_{\tau_1}\right] &= \mathbb{E}\left[\int_{\tau_1}^{\tau_2} [SN(V)_{s-} - U_{s-}] dA_s / \mathcal{F}_{\tau_1}\right] \\ &\quad - \mathbb{E}\left[\int_{\tau_1}^{\tau_2} [SN(V)_{s-} - V_{s-}] dA_s / \mathcal{F}_{\tau_1}\right] \\ &\geq 0.\end{aligned}$$

To conclude the proof, we need the uniqueness of the solution. Let U, $U' \in K \cap H^2$ two solutions of SVI (1.13). Then

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} (U_{s-} - U'_{s-}) d(U_s - U'_s) / \mathcal{F}_{\tau_1}\right] \geq 0. \tag{1.14}$$

Using Itô formula on $(U - U')^2$ leads to

$$(U_{\tau_2} - U'_{\tau_2})^2 - (U_{\tau_1} - U'_{\tau_1})^2 = 2 \int_{\tau_1}^{\tau_2} (U_{s-} - U'_{s-}) d(U_s - U'_s) + [U - U']_{\tau_2} - [U - U']_{\tau_1}.$$

Taking the conditional expectation leads to

$$\mathbb{E}\left[(U_{\tau_2} - U'_{\tau_2})^2 / \mathcal{F}_{\tau_1}\right] - (U_{\tau_1} - U'_{\tau_1})^2 \geq 0.$$

At last, we have for $\tau_2 = T$

$$-(U_{\tau_1} - U'_{\tau_1})^2 \geq 0.$$

□

1.3 A priori estimate

In this subsection, we provide an interesting a priori estimate, mainly based on the above characterization of strong envelop. On an arbitrary stochastic interval $(\sigma_1, \sigma_2]$, we are going to formulate and prove a priori estimate for the increment of the predictable process A component of strong envelop.

Theorem 1.5 *Let X be a bounded progressively measurable process, and U its strong envelop . Then, for any càdlàg process X^* such that*

$$X \leq X^* \leq U, \qquad a.s.,\ a.e.$$

we have the following inequality for the increasing process k component of the strong envelop U:

$$\mathbb{E}\Big(A_{\sigma_2} - A_{\sigma_1}/F_{\sigma_1}\Big) \leq \mathbb{E}\Big(X_{\tau^{\varepsilon}_{\sigma_1}\wedge\sigma_2} - X_{\sigma_2}/F_{\sigma_1}\Big) + \varepsilon \tag{1.15}$$

where $\tau^{\varepsilon}_{\sigma_1}$ is defined in Theorem 1.3 and $\varepsilon > 0$. And for any $p \geq 1$:

$$\|A_{\sigma_2} - A_{\sigma_1}\|_{L^p} \leq p\Big\| \sup_{\sigma_1 \leq s \leq \sigma_2} |X_{\sigma_2} - X_s|\Big\|_{L^p}. \tag{1.16}$$

Proof. The proof is done by the same techniques in Theorem 2.2 in [1], so we omit it. □

In the following, we adapt Theorem 2.3 in [1] to our setting, without providing the proof, since it's similar.

Theorem 1.6 *Let U^i, i=1, 2, be the strong envelop of X^i with $X^i \in L^2_{\mathcal{F}}(0,T)$. If there exist $X^i \in S^2$ such that $U^i \leq X^i \leq z^i$, then for any arbitrary stochastic interval $(\tau_1, \tau_2]$ the following estimate hold:*

$$\begin{aligned}&\mathbb{E}\Big((U^2_{\tau_1} - U^1_{\tau_1})^2 + [U^2 - U^1]_{\tau_2} - [U^2 - U^1]_{\tau_1}\Big)\\ &\leq 4\Big\| \sup_{\tau_1 \leq t \leq \tau_2} |X^2_t - X^1_t|\Big\|_{L^2} \cdot \left(\Big\| \sup_{\tau_1 \leq t \leq \tau_2} |X^1_t - X^1_{\tau_2}|\Big\|_{L^2} + \Big\| \sup_{\tau_1 \leq t \leq \tau_2} |X^2_t - X^2_{\tau_2}|\Big\|_{L^2}\right).\end{aligned}$$

In particular

$$\begin{aligned}\mathbb{E}[U^2 - U^1]_T \;\;\leq\;\; & 4\Big\| \sup_{0 \leq t \leq T} |X^2_t - X^1_t|\Big\|_{L^2}\\ & \times\left(\Big\| \sup_{0 \leq t \leq T} |X^1_t - X^1_T|\Big\|_{L^2} + \Big\| \sup_{t \geq 0} |X^2_t - X^2_T|\Big\|_{L^2}\right) + \mathbb{E}(X^2_T - X^1_T)^2.\end{aligned}$$

2 Conclusions

The strong envelop is an important tool as in optimal stopping theory, this notion generalize the one of Snell envelop. We provide a characterizations of strong envelop by two way: First, it can be characterized by general Skorohod equation. Second, we show that is the unique solution of the SVI. This notion could be applied in hedging problem of American option, backward stochastic differential equation with irregular lower barrier in infinite time horizon, as well as in switching problem.

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